

# AN APPROACH TO COMPLETE CONVERGENCE THEOREMS FOR DEPENDENT RANDOM FIELDS VIA APPLICATION OF FUK-NAGAEV INEQUALITY

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**Abstract:** Let  $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$  be a random field i.e. a family of random variables indexed by  $\mathbb{N}^d$ ,  $d \geq 2$ . Complete convergence, convergence rates for non identically distributed, negatively dependent and martingale random fields are studied by application of Fuk-Nagaev inequality. The results are proved in asymmetric convergence case i.e. for the norming sequence equal  $n_1^{\alpha_1} \cdot n_2^{\alpha_2} \cdot \dots \cdot n_d^{\alpha_d}$ , where  $(n_1, n_2, \dots, n_d) = \mathbf{n} \in \mathbb{N}^d$  and  $\min_{1 \leq i \leq d} \alpha_i \geq \frac{1}{2}$ .

**Keywords:** Baum-Katz type theorems, complete convergence, negatively dependent random fields, martingale random fields, Fuk Nagaev inequality.

**Mathematics Subject Classification:** 60F15

## 1 Introduction.

We observe again an interest in complete convergence theorems, which are discussed for weighted sum of dependent random variables, sums of random numbers of random variables, arrays of random variables or random fields.

We will consider random variables on probability space  $(\Omega, \mathfrak{F}, P)$ , indexed by lattice points, i.e. by index set  $\mathbb{N}^d$ ,  $d \geq 2$ . The elements of  $\mathbb{N}^d$  denote:  $\mathbf{m} = (m_1, m_2, \dots, m_d)$ ,  $\mathbf{n} = (n_1, n_2, \dots, n_d)$  etc., we order them by coordinate wise ordering:  $\mathbf{m} \leq \mathbf{n}$  iff  $m_i \leq n_i, i = 1, 2, \dots, d$ . We mean  $\mathbf{n} \rightarrow +\infty$  iff  $\min_{1 \leq i \leq d} n_i \rightarrow +\infty$ . A family of random variables  $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$  we also call a random field, furthermore denote  $S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}$ .

This article is inspired by paper of Gut and Stadtmüller [6], where authors have studied Baum-Katz type theorems and obtained very general results for fields of independent identically distributed random variables while the normalizing sequence depends on different powers of different coordinates, i.e. they have studied convergence of the sums

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r - 2} P(\max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{k}}| > |\mathbf{n}^{\alpha}| \varepsilon), \quad (1)$$

where  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in (\frac{1}{2}, 1)^d$ , coordinates  $\alpha_i$  are arranged in non-decreasing order,  $\alpha_1 r \geq 1$  and  $|\mathbf{n}^{\alpha}| = n_1^{\alpha_1} \cdot \dots \cdot n_d^{\alpha_d}$  or the case of convergence of

$$\sum_{\mathbf{n}} |\mathbf{n}|^{(r/2) - 2} P(\max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{n}}| \geq \sqrt{\prod_{i=1}^p n_i \log(\prod_{i=1}^p n_i)} \prod_{i=p+1}^d n_i^{\alpha_i} \varepsilon), \quad (2)$$

where  $\alpha_1 = \frac{1}{2}$ ,  $p = \max\{k : \alpha_k = \alpha_1\}$  and  $r \geq 2$ .

In the cited paper of Gut and Stadtmüller [6] one can find the review and the comments of the so far obtained results and further references. The crucial step in the proofs of the above mentioned theorems is based on symmetrization/desymmetrization and Kahane-Hoffmann-Jørgensen (K-H-J) inequality. K-H-J inequality is very sharp but strictly connected to independence of random variables. In the proofs of Baum-Katz type theorems such a strong inequality is not needed, we can apply weaker one with an attribute of K-H-J

inequality and at same time valid for dependent random variables. Fuk-Nagaev inequality is playing essential role in the proof of such inequalities. Thus, by that approach, we are going to extend or give a compliments of some results of Peligrad [11], Gut et al.[6], [7], Kuczmaszewska et al. [9]. Also, we will be able to extend the results of Ghosal et al. [5], Sung [15], Dehua et al [8] to the random fields. Our result for martingale random field seems be a little bit more general even in one dimension case ( $d = 1$ ) than the following result of Ghosal and Chandra (cf. Theorem 1(b) and Theorem 2 of [5])

**Theorem 1.1.** *Let  $\{(X_{nk}, \mathfrak{F}_{nk}), k \geq 1\}$  be a sequence of square integrable martingale differences. Suppose, that there exist constants  $(M_n)$  such that  $\sum_{k=1}^{\infty} E(X_{nk}^2 | \mathfrak{F}_{n,k-1}) \leq M_n$  a.s., where  $\mathfrak{F}_{n0}$  is trivial for all  $n$ . Let  $(c_n)$  be a sequence of nonnegative numbers satisfying  $\sum_{n=1}^{\infty} c_n M_n^\lambda < \infty$  for some  $\lambda > 0$  and*

$$\bigwedge_{\varepsilon > 0} \sum_{n=1}^{\infty} c_n \sum_{k=1}^{\infty} P(|X_{nk}| > \varepsilon) < \infty \quad , \quad (3)$$

then

$$\bigwedge_{\varepsilon > 0} \sum_{n=1}^{\infty} c_n \sum_{k=1}^{\infty} P\left(\sup_{k \geq 1} \left| \sum_{i=1}^k X_{ni} \right| > \varepsilon\right) < \infty \quad . \quad (4)$$

In order to formulate our main results we recall some definitions.

**Definition 1.2.** *A finite family of random variables  $\{X_j, 1 \leq j \leq n\}$  is said to be negatively dependent (ND) if*

$$P\left[\bigcap_{j \leq n} (X_j \leq x_j)\right] \leq \prod_{j \leq n} P(X_j \leq x_j)$$

and

$$P\left[\bigcap_{j \leq n} (X_j > x_j)\right] \leq \prod_{j \leq n} P(X_j > x_j)$$

for  $x_i \in \mathbb{R}$ ,  $\mathbf{i} \leq \mathbf{n}$

An infinite family is ND if every finite subfamily is ND.

**Definition 1.3.** *The family of random variables  $\{X_{\mathbf{j}}, \mathbf{j} \in \mathbb{N}^d\}$  is said to be negatively associated (NA) if*

$$\text{cov}(f(X_{\mathbf{j}}, \mathbf{j} \in S), g(X_{\mathbf{i}}, \mathbf{i} \in T)) \leq 0$$

*for every disjoint subset  $S, T \subset \mathbb{N}^d$ , where  $f(X_{\mathbf{j}}, \mathbf{j} \in S)$  and  $g(X_{\mathbf{i}}, \mathbf{i} \in T)$  are coordinatwise increasing functions and the covariance exist.*

An infinite family is NA if every finite subfamily is NA.

Since, we are going to prove results for non-identically distributed random variables, the following conditions allow us to formulate them in simple form as in i.i.d. case.

**Definition 1.4.** *Random variables  $\{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$  are weakly mean bounded (WMB) by random variable  $\xi$  (possibly defined on different probability space) iff there exist some constants  $\kappa_1, \kappa_2 > 0$ ,  $\mathbf{n}_0 \in \mathbb{N}^d$  and  $x_0 > 0$  such that for every  $x > x_0$  and  $\mathbf{n} \geq \mathbf{n}_0$ ,  $\mathbf{n} \in \mathbb{N}^d$*

$$\kappa_2 \cdot P(|\xi| > x) \leq \frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \leq \mathbf{n}} P(|X_{\mathbf{k}}| > x) \leq \kappa_1 \cdot P(|\xi| > x)$$

If only the right hand side inequality is satisfied, we say that the random field  $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$  and the random variable  $\xi$  satisfy weak mean dominance (WMD) condition. WMB condition seems to be very natural one and not

very restricted, e.g. regular cover condition (cf. Pruss [12]) uses in the same context is much stronger and obviously implies weak mean bounded condition.

In Section 4 we will consider martingale random field, thus introduce the fundamental notions. Let  $\{\mathfrak{F}_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$  be a filtration of  $\sigma$ -algebras i.e.

$$(F1) \quad \text{if } \mathbf{k} \leq \mathbf{n} \Rightarrow \mathfrak{F}_{\mathbf{k}} \subset \mathfrak{F}_{\mathbf{n}} \subset \mathfrak{F}$$

An integrable family of random variables  $\{Z_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$ , adapted to  $\{\mathfrak{F}_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$  is called martingale random field, iff

$$\bigwedge_{\mathbf{k} \leq \mathbf{n}} E(Z_{\mathbf{n}} | \mathfrak{F}_{\mathbf{k}}) = Z_{\mathbf{k}} \quad \text{a.s.}$$

Let us observe, that for martingale  $\{(Z_{\mathbf{n}}, \mathfrak{F}_{\mathbf{n}}), \mathbf{n} \in \mathbb{N}^d\}$

$$X_{\mathbf{n}} = \sum_{\mathbf{a} \in \{0,1\}^r} (-1)^{\sum_{i=1}^r a_i} Z_{\mathbf{n}-\mathbf{a}},$$

where  $\mathbf{a} = (a_1, a_2, \dots, a_r)$  and  $\mathbf{n} \in \mathbb{N}^d$ , are martingale differences with respect to  $\{\mathfrak{F}_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ .

## 2 Auxiliary Lemmas

Let  $\{a_{\mathbf{k},\mathbf{n}}, \mathbf{k}, \mathbf{n} \in \mathbb{N}^d\}$  be a family of real numbers, such that  $0 \leq a_{\mathbf{k},\mathbf{n}} < 1$ , then we have.

**Lemma 2.1.** *If  $\sum_{\mathbf{k} \leq \mathbf{n}} a_{\mathbf{k},\mathbf{n}} \rightarrow 0$  as  $\mathbf{n} \rightarrow \infty$ , then for a given  $0 < \delta < 1$  and  $\mathbf{n}$  sufficiently large*

$$1 - \prod_{\mathbf{k} \leq \mathbf{n}} (1 - a_{\mathbf{k},\mathbf{n}}) \geq (1 - \delta) \sum_{\mathbf{k} \leq \mathbf{n}} a_{\mathbf{k},\mathbf{n}}.$$

**Proof.** Let, for a given  $\delta$ ,  $\mathbf{n}$  be sufficiently large such that  $\sum_{\mathbf{k} \leq \mathbf{n}} a_{\mathbf{k}, \mathbf{n}} \leq \delta(1-\delta)$ .

Thus

$$\begin{aligned} \prod_{\mathbf{k} \leq \mathbf{n}} (1 - a_{\mathbf{k}, \mathbf{n}}) &= \exp\left\{\sum_{\mathbf{k} \leq \mathbf{n}} \ln(1 - a_{\mathbf{k}, \mathbf{n}})\right\} \leq \exp\left\{-\sum_{\mathbf{k} \leq \mathbf{n}} a_{\mathbf{k}, \mathbf{n}}\right\} \leq \\ 1 - \sum_{\mathbf{k} \leq \mathbf{n}} a_{\mathbf{k}, \mathbf{n}} + \left(\sum_{\mathbf{k} \leq \mathbf{n}} a_{\mathbf{k}, \mathbf{n}}\right)^2 &\leq 1 - (1 - \delta) \sum_{\mathbf{k} \leq \mathbf{n}} a_{\mathbf{k}, \mathbf{n}} \end{aligned} \quad (5)$$

Now, assertion easily follows.  $\blacksquare$

The next lemma is simply consequence of WMB condition and the well known fact, that for any random variable  $X$  with  $E | X |^s < \infty$

$$E | X |^s = s \int_0^\infty x^{s-1} P[| X | > x] dx.$$

For some  $a > 0$ , let us put

$$X'_i = X_i I[| X_i | \leq a], \quad X''_i = X_i I[| X_i | > a],$$

and

$$\xi' = \xi I[| \xi | \leq a], \quad \xi'' = \xi I[| \xi | > a].$$

**Lemma 2.2.** *Let  $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$  be a field of random variables satisfying WMB condition with random variable  $\xi$  and constants  $\kappa_1, \kappa_2$ . Let  $s > 0$ .*

$$(a) \text{ If } E | \xi |^s < \infty, \text{ then } \kappa_2 E | \xi |^s \leq \frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \leq \mathbf{n}} E | X_{\mathbf{k}} |^s \leq \kappa_1 E | \xi |^s.$$

$$(b) \kappa_2 E | \xi' |^s \leq \frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \leq \mathbf{n}} E | X'_{\mathbf{k}} |^s \leq \kappa_1 E | \xi' |^s.$$

$$(c) \kappa_2 E | \xi'' |^s \leq \frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \leq \mathbf{n}} E | X''_{\mathbf{k}} |^s \leq \kappa_1 E | \xi'' |^s.$$

The following properties of ND random variables, proved by Bozorgnia et al. [2], for sequences of r.v., obviously hold true for ND random fields.

**Lemma 2.3.** *Let  $\{X_{\mathbf{k}}, \mathbf{k} \leq \mathbf{n}\}$  be a field of ND random variables and  $\{f_{\mathbf{k}}, \mathbf{k} \leq \mathbf{n}\}$  a family of Borel functions, which all are non-decreasing (non-increasing), then*

- (a)  *$\{f(X_{\mathbf{k}}), \mathbf{k} \leq \mathbf{n}\}$  is a ND random field,*  
(b) *if additionally,  $X_{\mathbf{k}}$  are non-negative, we have*

$$E \left( \prod_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}} \right) \leq \prod_{\mathbf{k} \leq \mathbf{n}} EX_{\mathbf{k}}.$$

**Lemma 2.4.** *Assume, that  $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$  is a field of zero mean, square integrable ND random variables WMD by random variable  $\xi$  and such that  $E\xi^2 = \sigma^2 < \infty$ , then*

(a) 
$$E \left( \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}} \right)^2 \leq \kappa_1 \sigma^2 |\mathbf{n}|,$$

*if additionally  $P(X_{\mathbf{k}} \leq b) = 1$  for every  $\mathbf{k} \leq \mathbf{n}$ , then*

(b) 
$$P \left( \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}} > x \right) \leq e^{-tx + \kappa_1 \sigma^2 |\mathbf{n}|}$$

*for all  $x, b > 0$  and  $0 < t < \frac{1}{b}$ .*

**Proof.** By lemma 2.3 and 2.2a one can obtain

$$E \left( \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}} \right)^2 \leq \sum_{\mathbf{k} \leq \mathbf{n}} EX_{\mathbf{k}}^2 \leq \kappa_1 \sigma^2 |\mathbf{n}|,$$

thus (a) holds. Standard inequalities:  $e^x \leq 1+x+x^2$  for  $0 < x < 1$ ,  $1+x < e^x$  Lemma 2.3 and (a) lead us to inequality (b). ■

Let us put  $M_{\mathbf{n}}^r = \sum_{\mathbf{k} \leq \mathbf{n}} E |X_{\mathbf{k}}|^r$ ,  $\lambda_{\mathbf{k}} = EX_{\mathbf{k}} I[X_{\mathbf{k}} \geq -y]$  and  $\Lambda_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} \lambda_{\mathbf{k}, y}$

**Lemma 2.5.** *Assume that  $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$  be a field of zero mean ND random variables with finite an absolute  $r$ -th moment,  $1 \leq r \leq 2$ , then exist constant*

$C > 0$  such that for every  $x > 0$  and  $j > 0$

$$P(|S_{\mathbf{n}}| > x) \leq P(\max_{\mathbf{k} \leq \mathbf{n}} |X_{\mathbf{k}}| > \frac{x}{j}) + C(\frac{1}{x^r} M_{\mathbf{n}}^r)^j$$

**Proof.** Fakoor et al. have proved Fuk-Nagaev inequality for sequences ND random variables, Theorem 3 of [4], since the proof doesn't involve the order of index set, inequality holds true for  $d \geq 2$  case, thus under assumption for any  $y > 0$  we have

$$\begin{aligned} P(|S_{\mathbf{n}}| > x) &\leq P(\max_{\mathbf{j} \leq \mathbf{n}} |X_{\mathbf{j}}| > y) + \\ 2 \exp\left\{\frac{x}{y} - \left(\frac{x}{y} - \frac{\Lambda_{\mathbf{n}}}{y} + \frac{M_{\mathbf{n}}^r}{y^r}\right) \ln\left(1 + \frac{xy^{r-1}}{M_{\mathbf{n}}^r}\right)\right\} &= I_1 \end{aligned} \quad (6)$$

Since for all  $\mathbf{k} \in \mathbb{N}^d$

$$\frac{\lambda_{\mathbf{k},y}}{y} = \frac{E |X_{\mathbf{k}}| I[|X_{\mathbf{k}}| \geq y]}{y} \leq \frac{E |X_{\mathbf{k}}|^r}{y^r},$$

thus putting  $\frac{x}{y} = j$ , we obtain

$$\begin{aligned} I_1 &\leq P(\max_{\mathbf{k} \leq \mathbf{n}} |X_{\mathbf{k}}| > \frac{x}{j}) + 2 \exp\left\{j - j \ln\left(1 + \frac{x^r j^{1-r}}{M_{\mathbf{n}}^r}\right)\right\} \leq \\ P(\max_{\mathbf{k} \leq \mathbf{n}} |X_{\mathbf{k}}| > \frac{x}{j}) &+ 2e^j j^{(r-1)j} \left(\frac{M_{\mathbf{n}}^r}{x^r}\right)^j, \end{aligned}$$

which finishes the proof of lemma. ■

**Lemma 2.6.** *Let  $\xi$  be a random variable such that  $E |\xi|^{\frac{1}{\alpha_1}} (\log_+ |\xi|)^{p-1} < \infty$ , then under our setting with  $\alpha_1 > \frac{1}{2}$*

$$\sum_{\mathbf{n} \in \mathbb{N}^d} \frac{E |\xi|^2 I[|\xi| \leq |\mathbf{n}^\alpha|]}{|\mathbf{n}^\alpha|^2} < \infty.$$



**Proof.** For every  $\nu \in \mathbf{N}$  define

$$\Delta f(\nu) = \text{card}\{(n_1, n_2, \dots, n_p) : n_1 \cdot n_2 \cdot \dots \cdot n_p = \nu\}.$$

From the proof of Theorem 2.1 by Gut et al. [7] one can deduce

$$\begin{aligned} \sum_{\mathbf{n} \in \mathbb{N}^d} \frac{E |\xi|^2 I[|\xi| \leq |\mathbf{n}^\alpha|]}{|\mathbf{n}^\alpha|^2} = \\ \sum_{\nu=1}^{\infty} \sum_{n_{p+1}, \dots, n_d=1}^{\infty} \Delta f(\nu) \frac{1}{\nu^{2\alpha_1} \cdot n_{p+1}^{2\alpha_{p+1}} \cdot \dots \cdot n_d^{2\alpha_d}} \sum_{j=1}^{\nu^{\alpha_1} \cdot n_{p+1}^{\alpha_{p+1}} \cdot \dots \cdot n_d^{\alpha_d}} E \xi^2 I[j-1 < \xi \leq j] \leq \\ CE |\xi|^{\frac{1}{\alpha_1}} (\log_+ |\xi|)^{p-1} < \infty, \end{aligned}$$

where  $C > 0$  is suitable constant. ■

### 3 Baum-Katz type theorems for ND random fields

The first two theorems of this Section are extensions and compliments of some results of Peligrad [11], Gut et al.[6], [7], Kuczmaszewska et al. [9].

**Theorem 3.1.** *Let  $r \geq 1$ ,  $\alpha_1 \geq \frac{1}{2}$ ,  $\alpha_1 r \geq 1$  and  $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$  be a zero mean random field of ND random variables, weak mean bounded by  $\xi$ . If*

$$E |\xi|^r (\log_+ |\xi|)^{p-1} < \infty, \quad (7)$$

then

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r - 2} P(|S_{\mathbf{n}}| > |\mathbf{n}^\alpha| \varepsilon) < \infty \quad \text{for all } \varepsilon > 0. \quad (8)$$

Conversly if

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r - 2} P(\max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{k}}| > |\mathbf{n}^\alpha| \varepsilon) < \infty \quad \text{for all } \varepsilon > 0, \quad (9)$$

then (7) holds.

**Proof.** (7)  $\Rightarrow$  (8). The general idea of the proof is based on the proof of Theorem 4.1 by Gut and Stadmüller [6], thus we sketch the proof showing differences. At the beginning, assume that  $\alpha_1 > \frac{1}{2}$ ,  $\alpha_1 r > 1$  and (7) holds. Applying Lemma 2.5 one can obtain

$$\begin{aligned}
& \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r - 2} P(|S_{\mathbf{n}}| > |\mathbf{n}^{\alpha}| \varepsilon) \leq \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r - 2} \sum_{\mathbf{k} \leq \mathbf{n}} P(|X_{\mathbf{k}}| > y) + \\
& + \frac{C}{\varepsilon^{rj}} \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r - 2} |\mathbf{n}^{\alpha}|^{-jr} \left( \sum_{\mathbf{k} \leq \mathbf{n}} E |X_{\mathbf{k}}|^r \right)^j \leq \\
& C_1 \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r - 1} P(|\xi| > |\mathbf{n}^{\alpha}| \varepsilon') + C_2 \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r - 2 + j} |\mathbf{n}^{\alpha}|^{-jr} (E |\xi|^r)^j \\
& = I_2 + I_3, \quad \text{where } \varepsilon' = \frac{\varepsilon}{j}.
\end{aligned} \tag{10}$$

The first sum  $I_2$  is finite by Lemma 2.2 of [6], the second one is estimated as follows

$$\begin{aligned}
I_2 & \leq C \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r - 2 + j} |\mathbf{n}^{\alpha}|^{-jr} \leq \\
& C \sum_{\mathbf{n}} \prod_{i=1}^d n_i^{\alpha_1 r - 2 + (1 - \alpha_1 r)j} \leq C \prod_{i=1}^d \sum_{n_i=1}^{\infty} n_i^{\alpha_1 r - 2 + (1 - \alpha_1 r)j} < \infty,
\end{aligned} \tag{11}$$

since exponent in the last sum can be less than  $(-1)$ , for  $j$  sufficiently large.

Now, assume that  $\alpha_1 > \frac{1}{2}$ ,  $\alpha_1 r = 1$ . Let  $Y_{\mathbf{k}, \mathbf{n}} = \min(|\mathbf{n}^{\alpha}|, |X_{\mathbf{k}}|) \operatorname{sgn}(X_{\mathbf{k}})$ ,  $X_{\mathbf{k}, \mathbf{n}} = X_{\mathbf{k}} I[|X_{\mathbf{k}}| \leq |\mathbf{n}^{\alpha}|]$  and  $T_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} Y_{\mathbf{k}, \mathbf{n}}$ . Thus we get

$$\begin{aligned}
& \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P(|S_{\mathbf{n}}| > 2|\mathbf{n}^{\alpha}| \varepsilon) \leq \\
& \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P(|T_{\mathbf{n}}| > |\mathbf{n}^{\alpha}| \varepsilon) + \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P(|S_{\mathbf{n}} - T_{\mathbf{n}}| > |\mathbf{n}^{\alpha}| \varepsilon) = I_4 + I_5
\end{aligned} \tag{12}$$

The first sum can be estimated by applying Chebyshev inequality, Lemma 2.4 and 2.2, WMD condition consecutively:

$$\begin{aligned}
I_4 & \leq \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \frac{E(T_{\mathbf{n}} - ET_{\mathbf{n}})^2}{\varepsilon^2 |\mathbf{n}^{\alpha}|^2} \leq C \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \frac{ET_{\mathbf{n}}^2}{|\mathbf{n}^{\alpha}|^2} \leq \\
& C \left( \sum_{\mathbf{n}} \left[ \frac{1}{|\mathbf{n}|} \frac{\sum_{\mathbf{k} \leq \mathbf{n}} EX_{\mathbf{k},\mathbf{n}}^2}{|\mathbf{n}^{\alpha}|^2} + \frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \leq \mathbf{n}} P(|X_{\mathbf{k}}| > |\mathbf{n}^{\alpha}|) \right] \right) \leq \\
& C \left( \sum_{\mathbf{n}} \frac{E|\xi|^2 I[|\xi| \leq |\mathbf{n}^{\alpha}|]}{|\mathbf{n}^{\alpha}|^2} + \sum_{\mathbf{n}} P(|\xi| > |\mathbf{n}^{\alpha}|) \right) \leq CE |\xi|^{\frac{1}{\alpha_1}} (\log_+ |\xi|)^{p-1}.
\end{aligned} \tag{13}$$

The last inequality follows from Lemma 2.6 and Lemma 2.2 of [6] respectively.

On the other hand

$$\begin{aligned}
I_5 & \leq \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P\left(\sum_{\mathbf{k} \leq \mathbf{n}} |X_{\mathbf{k}}| I[|X_{\mathbf{k}}| > |\mathbf{n}^{\alpha}|] > \varepsilon |\mathbf{n}^{\alpha}|\right) \leq \\
& \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P\left(\sum_{\mathbf{k} \leq \mathbf{n}} |X_{\mathbf{k}}| > |\mathbf{n}^{\alpha}|\right) \leq C \sum_{\mathbf{n}} P(|\xi| > |\mathbf{n}^{\alpha}|) < \infty,
\end{aligned} \tag{14}$$

by WMD condition and Lemma 2.2 of [6].

The implication (9)  $\Rightarrow$  (7). Firstly, let us observe, that the negative and positive part of ND random variables are still ND. Thus

$$\begin{aligned}
P(\max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{k}}| > \varepsilon \mid \mathbf{n}^\alpha \mid \varepsilon) &\geq P(\max_{\mathbf{k} \leq \mathbf{n}} |X_{\mathbf{k}}| > 2 \mid \mathbf{n}^\alpha \mid \varepsilon) \geq \\
P(\max_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}^+ > 2 \mid \mathbf{n}^\alpha \mid \varepsilon) &= 1 - P(\bigcap_{\mathbf{k} \leq \mathbf{n}} [X_{\mathbf{k}}^+ \leq 2 \mid \mathbf{n}^\alpha \mid \varepsilon]) \geq \\
1 - \prod_{\mathbf{k} \leq \mathbf{n}} P(X_{\mathbf{k}}^+ \leq 2 \mid \mathbf{n}^\alpha \mid \varepsilon) &= 1 - \prod_{\mathbf{k} \leq \mathbf{n}} (1 - P(X_{\mathbf{k}}^+ > 2 \mid \mathbf{n}^\alpha \mid \varepsilon))
\end{aligned} \tag{15}$$

From (9) and (15) it's easy to see, that  $\prod_{\mathbf{k} \leq \mathbf{n}} (1 - P(X_{\mathbf{k}}^+ > 2\varepsilon \mid \mathbf{n}^\alpha \mid \varepsilon)) \rightarrow 1$  as  $\mathbf{n} \rightarrow \infty$ , what is equivalent to

$$\sum_{\mathbf{k} \leq \mathbf{n}} P(X_{\mathbf{k}}^+ > 2\varepsilon \mid \mathbf{n}^\alpha \mid \varepsilon) \rightarrow 0 \quad \text{as } \mathbf{n} \rightarrow \infty. \tag{16}$$

Analogously, we can get

$$\sum_{\mathbf{k} \leq \mathbf{n}} P(X_{\mathbf{k}}^- > 2\varepsilon \mid \mathbf{n}^\alpha \mid \varepsilon) \rightarrow 0 \quad \text{as } \mathbf{n} \rightarrow \infty. \tag{17}$$

Now, applying Lemma 2.1 with  $a_{\mathbf{k}, \mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} P(X_{\mathbf{k}}^+ > \varepsilon \mid \mathbf{n}^\alpha \mid \varepsilon)$  and  $a_{\mathbf{k}, \mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} P(X_{\mathbf{k}}^- > \varepsilon \mid \mathbf{n}^\alpha \mid \varepsilon)$ , WMB condition and Lemma 2.2 of [6], we have

$$\begin{aligned}
\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r - 2} P(\max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{k}}| > \varepsilon \mid \mathbf{n}^\alpha \mid \varepsilon) &\geq C_1 \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r - 2} P(\max_{\mathbf{k} \leq \mathbf{n}} |X_{\mathbf{k}}| > 2\varepsilon \mid \mathbf{n}^\alpha \mid \varepsilon) \geq \\
C_2 \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r - 2} \sum_{\mathbf{k} \leq \mathbf{n}} P(|X_{\mathbf{k}}| > 2\varepsilon \mid \mathbf{n}^\alpha \mid \varepsilon) &\geq C_3 \sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r - 1} \sum_{\mathbf{k} \leq \mathbf{n}} P(|\xi| > 2\varepsilon \mid \mathbf{n}^\alpha \mid \varepsilon) \geq \\
C_4 E |\xi|^r (\log_+ |\xi|)^{p-1}.
\end{aligned}$$

(18)

■

**Theorem 3.2.** Let  $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$  be a field of zero mean ND random variables satisfying WMB condition with r.v.  $\xi$  and suppose, that  $r \geq 2$ ,  $\alpha_1 = \frac{1}{2}$ ,  $\alpha_1 r \geq 1$ . If

$$E |\xi|^r (\log_+ |\xi|)^{p-1-\frac{r}{2}} < \infty \quad \text{and} \quad E\xi^2 = \sigma^2 < \infty, \quad (19)$$

then

$$\sum_{\mathbf{n}} |\mathbf{n}|^{(r/2)-2} P(|S_{\mathbf{n}}| \geq \sqrt{\prod_{i=1}^p n_i \log(\prod_{i=1}^p n_i) \prod_{i=p+1}^d n_i^{\alpha_i} \varepsilon}) < \infty \quad (20)$$

for  $\varepsilon > \sigma_1 \sqrt{r-2}$ , where  $p = \max\{k : \alpha_k = \alpha_1\}$  and  $\sigma_1^2 = \kappa_1 \sigma^2$ .

Conversely, suppose either  $r = 2$  and  $p \geq 2$  or that  $r > 2$ .

If

$$\sum_{\mathbf{n}} |\mathbf{n}|^{(r/2)-2} P(\max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{n}}| \geq \sqrt{\prod_{i=1}^p n_i \log(\prod_{i=1}^p n_i) \prod_{i=p+1}^d n_i^{\alpha_i} \varepsilon}) < \infty \quad (21)$$

for some  $\varepsilon > 0$ , then

$$E |\xi|^r (\log_+ |\xi|)^{p-1-\frac{r}{2}} < \infty. \quad (22)$$

**Proof.** (19)  $\Rightarrow$  (20), the case  $\alpha_1 = \frac{1}{2}$ ,  $r = 2$ .

Applying Lemma 2.5 with second moment and by WMB condition, we have

$$\begin{aligned} & \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} P \left( |S_{\mathbf{n}}| > \sqrt{\prod_{i=1}^p n_i \log(\prod_{i=1}^p n_i) \prod_{i=p+1}^d n_i^{\alpha_i} \varepsilon} \right) \leq \\ & 2 \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \sum_{\mathbf{k} \leq \mathbf{n}} P \left( |X_{\mathbf{k}}| > \sqrt{\prod_{i=1}^p n_i \log(\prod_{i=1}^p n_i) \prod_{i=p+1}^d n_i^{\alpha_i} \varepsilon'} \right) + \\ & \frac{C}{\varepsilon^2 j} \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \left( \frac{\sum_{\mathbf{k} \leq \mathbf{n}} EX_{\mathbf{k}}^2}{\prod_{i=1}^p n_i \log(\prod_{i=1}^p n_i) (\prod_{i=p+1}^d n_i^{\alpha_i})^2} \right)^j \leq \end{aligned} \quad (23)$$

$$C_1 \sum_{\mathbf{n}} P \left( |\xi| > \sqrt{\prod_{i=1}^p n_i \log(\prod_{i=1}^p n_i)} \prod_{i=p+1}^d n_i^{\alpha_i} \varepsilon' \right) +$$

$$C_2 \sum_{\mathbf{n}} \frac{1}{|\mathbf{n}|} \left( \frac{|\mathbf{n}| E \xi^2}{\prod_{i=1}^p n_i \log(\prod_{i=1}^p n_i) \prod_{i=p+1}^d n_i^{2\alpha_i}} \right)^j = I_4 + I_5,$$

where  $C_1$  and  $C_2$  are suitable constants. The first sum is finite by assumption (19) and the second one, by the same arguments as in proof of Theorem 4.1 of [6].

The case  $\alpha_1 = \frac{1}{2}$ ,  $r > 2$ . Let  $0 < \eta < \alpha_{p+1} - \frac{1}{2}$  and  $\beta_i = \alpha_i - \eta$  for  $i = p+1, p+2, \dots, d$ . Furthermore, we set

$$a_{\mathbf{n}} = \sqrt{\prod_{i=1}^p n_i \log(\prod_{i=1}^p n_i)} \prod_{i=p+1}^d n_i^{\alpha_i}, \quad b_{\mathbf{n}} = \frac{2\delta\sigma_1^2}{\varepsilon} \sqrt{\frac{\prod_{i=1}^p n_i}{\log(\prod_{i=1}^p n_i)} \prod_{i=p+1}^d n_i^{\beta_i}}$$

$$c_{\mathbf{n}} = \delta a_{\mathbf{n}} \quad \text{and} \quad d_{\mathbf{n}} = \sqrt{\prod_{i=1}^p n_i \log(\prod_{i=1}^p n_i)} \prod_{i=p+1}^d n_i^{1-\beta_i}$$

Let us put  $X'_{\mathbf{k}} = X_{\mathbf{k}} I[X_{\mathbf{k}} < b_{\mathbf{n}}] + b_{\mathbf{n}} I[X_{\mathbf{k}} \geq b_{\mathbf{n}}]$  and  $S'_{\mathbf{k}} = \sum_{\mathbf{k} \leq \mathbf{n}} X'_{\mathbf{k}}$ .

Define the events:

$$A_{\mathbf{n}}^1 = \{S'_{\mathbf{n}} > \varepsilon a_{\mathbf{n}}\}, \quad A_{\mathbf{n}}^2 = \{\text{at least two } \mathbf{k}, \mathbf{k} \leq \mathbf{n} : b_{\mathbf{n}} < X_{\mathbf{k}} < c_{\mathbf{n}}\}$$

$$A_{\mathbf{n}}^3 = \{\text{at least one } \mathbf{k}, \mathbf{k} \leq \mathbf{n} : X_{\mathbf{k}} \geq c_{\mathbf{n}}\}. \quad \text{and} \quad A_{\mathbf{n}} = \{S_{\mathbf{n}} > (\varepsilon + 2\delta)a_{\mathbf{n}}\}.$$

It's clearly that  $A_{\mathbf{n}} \subset A_{\mathbf{n}}^1 \cup A_{\mathbf{n}}^2 \cup A_{\mathbf{n}}^3$  thus  $P(A_{\mathbf{n}}) \leq P(A_{\mathbf{n}}^1) + P(A_{\mathbf{n}}^2) + P(A_{\mathbf{n}}^3)$ .

We start from the estimation of  $P(A_{\mathbf{n}}^1)$ . The first step, since  $\{X_{\mathbf{k}}, \mathbf{k} \leq \mathbf{n}\}$  is

a field of zero mean random variables satisfying WMB condition, we have

$$\begin{aligned} |ES'_n| &\leq \sum_{\mathbf{k} \leq \mathbf{n}} EX_{\mathbf{k}} I[X_{\mathbf{k}} \geq b_{\mathbf{n}}] \leq \kappa_1 |\mathbf{n}| E\xi I[\xi \geq b_{\mathbf{n}}] \leq \\ &\frac{\kappa_1 |\mathbf{n}|}{b_{\mathbf{n}}} E\xi^2 I[\xi \geq b_{\mathbf{n}}] = o(d_{\mathbf{n}}) \end{aligned}$$

Further arguments and details are the same as proof of (4.4-4.6) of [6], hence

$$\sum_{\mathbf{n}} |\mathbf{n}|^{(r/2-2)} P(A_{\mathbf{n}}^1) < \infty \quad \text{for } \varepsilon > \sigma_1 \frac{1+\delta}{1-\delta} \sqrt{r-2} \quad \text{and all } \delta > 0. \quad (24)$$

In estimation of  $P(A_{\mathbf{n}}^2)$ , we exploit the ND and WMD property of  $\{X_{\mathbf{k}}, \mathbf{k} \leq \mathbf{n}\}$  and thereafter by the same manner as in the proof of (4.7) of [6]

$$P(A_{\mathbf{n}}^2) \leq \sum_{\mathbf{k} \leq \mathbf{n}} \sum_{\mathbf{l} \leq \mathbf{n}, \mathbf{l} \neq \mathbf{k}} P(X_{\mathbf{k}} > b_{\mathbf{n}}, X_{\mathbf{l}} > b_{\mathbf{n}}) \leq \kappa_1^2 |\mathbf{n}|^2 (P(\xi > b_{\mathbf{n}}))^2$$

thus

$$\begin{aligned} \sum_{\mathbf{n}} |\mathbf{n}|^{(r/2-2)} P(A_{\mathbf{n}}^2) &\leq \\ C(r, \delta) \sum_{\mathbf{n}} |\mathbf{n}|^{-r/2} \frac{\left( \log \left( \prod_{i=1}^p n_i \right) \right)^r}{(\log |\mathbf{n}|)^{2(p-1)-r}} &< \infty \quad \text{for all } \delta > 0. \end{aligned} \quad (25)$$

Finally, by Lemma 2.1(c) of [6]

$$\begin{aligned} \sum_{\mathbf{n}} |\mathbf{n}|^{(r/2-2)} P(A_{\mathbf{n}}^3) &\leq \sum_{\mathbf{n}} |\mathbf{n}|^{(r/2-2)} \sum_{\mathbf{k} \leq \mathbf{n}} P(X_{\mathbf{k}} \geq c_{\mathbf{n}}) \leq \\ \kappa_1 \sum_{\mathbf{n}} |\mathbf{n}|^{(r/2-1)} P(\xi \geq \delta \sqrt{\prod_{i=1}^p n_i \log(\prod_{i=1}^p n_i)} \prod_{i=p+1}^d n_i^{\alpha_i}) &< \infty \end{aligned} \quad (26)$$

Now, let us put  $X''_{\mathbf{k}} = X_{\mathbf{k}} I[X_{\mathbf{k}} > -b_{\mathbf{n}}] - b_{\mathbf{n}} I[X_{\mathbf{k}} \leq -b_{\mathbf{n}}]$ ,  $S''_{\mathbf{k}} = \sum_{\mathbf{k} \leq \mathbf{n}} X''_{\mathbf{k}}$ .

The events:

$$B_{\mathbf{n}}^1 = \{S_{\mathbf{n}}'' < -\varepsilon a_{\mathbf{n}}\}, \quad B_{\mathbf{n}}^2 = \{\text{at least two } \mathbf{k}, \mathbf{k} \leq \mathbf{n} : -c_{\mathbf{n}} < X_{\mathbf{k}} < -b_{\mathbf{n}}\}$$

$$B_{\mathbf{n}}^3 = \{\text{at least one } \mathbf{k}, \mathbf{k} \leq \mathbf{n} : X_{\mathbf{k}} \leq -c_{\mathbf{n}}\}. \quad \text{and } B_{\mathbf{n}} = \{S_{\mathbf{n}} < -(\varepsilon + 2\delta)a_{\mathbf{n}}\}$$

satisfy the inclusion  $B_{\mathbf{n}} \subset B_{\mathbf{n}}^1 \cup A_{\mathbf{n}}^2 \cup B_{\mathbf{n}}^3$ .

Likewise as the proof of (24), (25), (26), we can show that

$$\sum_{\mathbf{n}} |\mathbf{n}|^{(r/2-2)} P(B_{\mathbf{n}}) < \infty \quad (27)$$

By (24), (25), (26) and (27), eventually we have

$$\sum_{\mathbf{n}} |\mathbf{n}|^{(r/2-2)} P(|S_{\mathbf{n}}| > (\varepsilon + 2\delta)a_{\mathbf{n}}) < \infty \quad \text{for } \varepsilon > \sigma_1 \frac{1+\delta}{1-\delta} \sqrt{r-2} \quad \text{and all } \delta > 0.$$

Arbitrariness of  $\delta$ , allows us to conclude the implication (19)  $\Rightarrow$  (20). The implication (21)  $\Rightarrow$  (22) one can prove similarly as the implication (9)  $\Rightarrow$  (7). ■

At the end of this Section, we present one more result, which is an extension of some results of Sung [15] and Dehua et al [8], to ND random fields.

Suppose, that  $\{\mathbf{k}_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$  is a family of lattice points of  $\mathbb{N}^d$ .

**Theorem 3.3.** *Let  $\{X_{\mathbf{n},\mathbf{i}}, \mathbf{i} \leq \mathbf{k}_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$  be an array of rowwise ND random variables with  $EX_{\mathbf{n},\mathbf{i}} = 0$  and  $E|X_{\mathbf{n},\mathbf{i}}|^r < \infty$  for  $1 \leq r \leq 2$ ,  $\mathbf{i} \leq \mathbf{k}_{\mathbf{n}}$  and  $\mathbf{n} \in \mathbb{N}^d$ . Furthermore assume, that  $\{a_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$  is a sequence of nonnegative constants. If the following conditions hold:*

- $\sum_{\mathbf{n}} a_{\mathbf{n}} \sum_{\mathbf{i} \leq \mathbf{k}_{\mathbf{n}}} P(|X_{\mathbf{n},\mathbf{i}}| > \epsilon) < \infty$  for all  $\epsilon > 0$
- there exist  $j > 0$  such that  $\sum_{\mathbf{n}} a_{\mathbf{n}} \left( \sum_{\mathbf{i} \leq \mathbf{k}_{\mathbf{n}}} E|X_{\mathbf{n},\mathbf{i}}|^r \right)^j < \infty$ ,



then

$$\sum_{\mathbf{n}} a_{\mathbf{n}} P(| \sum_{\mathbf{i} \leq \mathbf{k}_{\mathbf{n}}} X_{\mathbf{n},\mathbf{i}} | > \epsilon) < \infty \quad \text{for all } \epsilon > 0.$$

**Proof.** By straightforward application of Lemma 2.5. ■

## 4 Martingale random fields

In introduction, we have given fundamental definition of martingale random field. It is known, that we can't obtain any sensible results for multi-parameter martingale without any additional conditions for the filtration. This brings us to commutation hypothesis - also known as (F4) and some others:

(F3)  $\mathfrak{F}_0$  contains all zero events of  $\mathfrak{F}$ ,

(F4)  $\bigwedge_{\mathbf{k}, \mathbf{n} \in \mathbb{N}^d}$  and any bounded,  $\mathfrak{F}_{\mathbf{k}}$ -measurable random variable  $Y$   
 $E(Y | \mathfrak{F}_{\mathbf{n}}) = E(Y | \mathfrak{F}_{\mathbf{n} \wedge \mathbf{k}})$  a.s.

The following notions help us to recall definition of strong martingale random field and j-martingale, which we exploit in this Section.

Let  $J \subseteq \{1, 2, \dots, d\}$ ,  $CJ = \{1, 2, \dots, d\} \setminus J$  and for a given  $(n_1, \dots, n_d) \in N^d$ , denote  $\mathfrak{F}_{\mathbf{n}}^J = \bigvee_{(n_j \in N, j \in CJ)} \mathfrak{F}_n$ . For example,  $\mathfrak{F}_{\mathbf{n}}^{1,2} = \bigvee_{n_3, \dots, n_d \in N} \mathfrak{F}_{\mathbf{n}}$ .

Thus, we can have equivalent form of (F4) condition (cf. Corollary 1 of [1]):

(F4') for any bounded random variable  $Y$

$$\bigwedge_{\mathbf{n} \in N^d} \bigwedge_{J \subseteq \{1, 2, \dots, d\}} E(Y | \mathfrak{F}_{\mathbf{n}}^J | \mathfrak{F}_{\mathbf{n}}^{CJ}) = E(Y | \mathfrak{F}_{\mathbf{n}}).$$

Let us put  $\mathcal{G}_{\mathbf{n}} = \bigvee_{j=1}^d \mathfrak{F}_{\mathbf{n}}^j$  and  $\tilde{\mathfrak{F}}_{\mathbf{n}-1} = \mathcal{G}_{\mathbf{n}-1} \wedge \mathfrak{F}_{\mathbf{n}}$ , where  $\mathbf{n} - \mathbf{1} = (n_1 - 1, n_2 - 1, \dots, n_d - 1)$

Furthermore, we need the following conditions:

(X1)  $X_{\mathbf{n}} = 0$  if  $|\mathbf{n}| = 0$ ,

(X2) the family of random variables  $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$  is measurable with respect to family of  $\sigma$ -algebras  $\{\mathfrak{F}_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ ,

(X3)  $E(X_{\mathbf{n}} | \tilde{\mathcal{F}}_{\mathbf{n}-1}) \leq 0$  for very  $\mathbf{n} \in N^d$ ,

(X3')  $E(X_{\mathbf{n}} | \tilde{\mathcal{F}}_{\mathbf{n}-1}) = 0$  for very  $\mathbf{n} \in N^d$ ,

(F5)  $E(Y | \mathfrak{F}_{\mathbf{n}} | \mathcal{G}_{\mathbf{n}-1}) = E(Y | \tilde{\mathfrak{F}}_{\mathbf{n}-1})$  for every  $\mathbf{n} \in N^d$  and any bounded random variables  $Y$ .

An integrable family of random variables  $\{(X_{\mathbf{n}}, \mathfrak{F}_{\mathbf{n}}), \mathbf{n} \in \mathbb{N}^d\}$  satisfying condition (X2) is:

- strong martingale differences iff  $E(X_{\mathbf{n}} | \mathcal{G}_{\mathbf{n}-1}) = 0$  a.s.,
- j-martingale differences iff  $(X_{\mathbf{n}}, \mathfrak{F}_{\mathbf{n}}^j)$  is a one parameter martingale differences with respect to coordinate  $n_j$ .

Fuk-Nagaev inequality for martingale random fields was proved by Lagodowski [10] in the case  $d = 2$  and extended to the case  $d \geq 2$  by Borodhikin [1], both authors have obtained theorems for the bounded second conditional moments. We complete these results to the arbitrary  $r$ -th conditional absolute moment,  $1 \leq r \leq 2$ .

Let us assume, that there exist fields of positive numbers  $\{b_{\mathbf{k}}^r, \mathbf{k} \in \mathbb{N}^d\}$ ,  $\{d_{\mathbf{k}} \mathbf{k} \in \mathbb{N}^d\}$ ,  $\{\tilde{\lambda}_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$  and  $\{\tilde{m}_{\mathbf{k}}^r, \mathbf{k} \in \mathbb{N}^d\}$  such that

$$\begin{aligned} E(|X_{\mathbf{n}}|^r | I[|X_{\mathbf{n}}| \leq y] | \mathfrak{F}_{\mathbf{n}-1}) &\leq b_{\mathbf{k}}^r, & E(X_{\mathbf{k}} I[X_{\mathbf{k}} > -y] | \mathfrak{F}_{\mathbf{n}-1}) &\leq d_{\mathbf{k}}, \\ E(|X_{\mathbf{n}}|^r | \mathfrak{F}_{\mathbf{n}-1}) &\leq \tilde{m}_{\mathbf{k}}^r, & E(|X_{\mathbf{k}}| I[|X_{\mathbf{k}}| > y] | \mathfrak{F}_{\mathbf{n}-1}) &\leq \tilde{\lambda}_{\mathbf{k}} \end{aligned} \quad (28)$$

for every  $\mathbf{k} \in \mathbb{N}^d$  and denote

$$B_{\mathbf{k}}^r = \sum_{\mathbf{k} \leq \mathbf{n}} b_{\mathbf{k}}^r, D_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}}, \widetilde{M}_{\mathbf{n}}^r = \sum_{\mathbf{k} \leq \mathbf{n}} \tilde{m}_{\mathbf{k}}^r \text{ and } \widetilde{\Lambda}_{\mathbf{k}} = \sum_{\mathbf{k} \leq \mathbf{n}} \tilde{\lambda}_{\mathbf{k}}$$

**Theorem 4.1.** *Suppose, that family of  $\sigma$ -algebras  $\{\mathfrak{F}_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$  satisfies condition (F1),(F3),(F4) and (F5) in the case  $d > 2$ , family of random variables  $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$  satisfies conditions (X1)-(X3), (28) and let  $x, y > 0$ ,  $1 \leq r \leq 2$ ,*

*then the following inequality holds*

$$\begin{aligned} P(\max_{\mathbf{k} \leq \mathbf{n}} S_{\mathbf{k}} \geq x) &\leq P(\max_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}} \geq y) \\ &+ e^{d-1} \exp \left\{ \frac{x}{y} - \left( \frac{x - D_{\mathbf{n}}}{y} + \frac{B_{\mathbf{n}}^r}{y^r} \right) \ln \left[ \frac{xy^{r-1}}{B_{\mathbf{n}}^r} + 1 \right] \right\}, \end{aligned} \quad (29)$$

*if we assume (X3') instead of (X3), we have*

$$\begin{aligned} P(\max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{k}}| \geq x) &\leq P(\max_{\mathbf{k} \leq \mathbf{n}} |X_{\mathbf{k}}| \geq y) \\ &+ 2e^{d-1} \exp \left\{ \frac{x}{y} - \left( \frac{x - \widetilde{\Lambda}_{\mathbf{n}}}{y} + \frac{\widetilde{M}_{\mathbf{n}}^r}{y^r} \right) \ln \left[ \frac{xy^{r-1}}{\widetilde{M}_{\mathbf{n}}^r} + 1 \right] \right\}. \end{aligned} \quad (30)$$

**Proof.** (sketch) Let us put

$$\begin{aligned} \widetilde{X}_{\mathbf{k}} &= X_{\mathbf{k}} I[X_{\mathbf{k}} \leq y], \quad \widetilde{S}_{\mathbf{k}} = \sum_{\mathbf{k} \leq \mathbf{n}} \widetilde{X}_{\mathbf{k}} \text{ and} \\ Z_{\mathbf{k}} &= \widetilde{X}_{\mathbf{k}} - E(\widetilde{X}_{\mathbf{k}} | \mathfrak{F}_{\mathbf{n}-1}), \quad T_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} Z_{\mathbf{k}}. \end{aligned}$$

Obviously, we have

$$P(\max_{\mathbf{k} \leq \mathbf{n}} S_{\mathbf{k}} \geq x) \leq P(\max_{\mathbf{k} \leq \mathbf{n}} \widetilde{S}_{\mathbf{k}} \geq x) + P(\max_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}} \geq y). \quad (31)$$

From (X3) implies, that  $Z_{\mathbf{k}} \geq \widetilde{X}_{\mathbf{k}}$  a.s. and since  $\alpha > 1$ ,  $h > 0$

$$P(\max_{\mathbf{k} \leq \mathbf{n}} \widetilde{S}_{\mathbf{k}} \geq x) \leq P(\max_{\mathbf{k} \leq \mathbf{n}} e^{\alpha h T_{\mathbf{k}}} \geq e^{\alpha h x}). \quad (32)$$

Let us observe, that  $\{(e^{\alpha h T_{\mathbf{k}}}, \widetilde{\mathfrak{F}}_{\mathbf{k}}), \mathbf{k} \leq \mathbf{n}\}$  is positive submartingale.

Denote  $\mathbf{k}(j) = (k_1, k_2, \dots, k_{j-1}, k_{j+1}, \dots, k_d)$  for  $\mathbf{k} \in \mathbb{N}^d$  and  $1 \leq j \leq d$ , thus  $\{\max_{\mathbf{k}(d) \leq \mathbf{n}(d)} e^{\alpha h T_{\mathbf{k}}}, 1 \leq k_d \leq n_d\}$  is positive d-sumbartingale with respect to  $\{\widetilde{\mathfrak{F}}_{\mathbf{k}}, \mathbf{k} \leq \mathbf{n}\}$ . By application of standard Doob inequality to d-submartingale and Doob inequality for submartingale random field, cf. Shorack et al. [14]

$$\begin{aligned} P(\max_{\mathbf{k} \leq \mathbf{n}} e^{\alpha h T_{\mathbf{k}}} \geq e^{\alpha h x}) &\leq e^{-\alpha h x} E \left( \max_{\mathbf{k}(d) \leq \mathbf{n}(d)} (e^{h T_{\mathbf{k}(d) n_d}})^{\alpha} \right) \\ &\leq \left( \frac{\alpha}{\alpha - 1} \right)^{\alpha(d-1)} e^{-\alpha h x} E e^{\alpha h T_{\mathbf{n}}}. \end{aligned} \quad (33)$$

Furthermore, we need estimations:

- $E(e^{\alpha h \widetilde{X}_{\mathbf{k}}} | \widetilde{\mathfrak{F}}_{\mathbf{k}-1}) = E(e^{\alpha h \widetilde{X}_{\mathbf{k}}} I[\widetilde{X}_{\mathbf{k}} < -y] | \widetilde{\mathfrak{F}}_{\mathbf{k}-1}) +$   
 $E(e^{\alpha h \widetilde{X}_{\mathbf{k}}} I[|\widetilde{X}_{\mathbf{k}}| \leq y] | \widetilde{\mathfrak{F}}_{\mathbf{k}-1}) = I_{10} + I_{11},$
- $I_{10} \leq E(I[X_{\mathbf{k}} < -y] | \widetilde{\mathfrak{F}}_{\mathbf{k}-1}),$
- $I_{11} \leq \frac{e^{\alpha h y - 1 - \alpha h y}}{y^2} E \left( X_{\mathbf{k}}^2 I[0 < |X_{\mathbf{k}}| \leq y] | \widetilde{\mathfrak{F}}_{\mathbf{k}-1} \right) + \alpha h E(X_{\mathbf{k}} I[|X_{\mathbf{k}}| \leq y] | \widetilde{\mathfrak{F}}_{\mathbf{k}-1}) +$   
 $E(I[|X_{\mathbf{k}}| \leq y] | \widetilde{\mathfrak{F}}_{\mathbf{k}-1}) \leq \frac{e^{\alpha h y - 1 - \alpha h y}}{y^r} E \left( |X_{\mathbf{k}}|^r I[0 < |X_{\mathbf{k}}| \leq y] | \widetilde{\mathfrak{F}}_{\mathbf{k}-1} \right) +$   
 $\alpha h E(X_{\mathbf{k}} I[|X_{\mathbf{k}}| \leq y] | \widetilde{\mathfrak{F}}_{\mathbf{k}-1}).$

Thus

$$\begin{aligned}
& E(e^{\alpha h Z_{\mathbf{k}}} \mid \tilde{\mathfrak{F}}_{\mathbf{k}-1}) \leq \\
& e^{-\alpha h E(X_{\mathbf{k}} I[X_{\mathbf{k}} \leq y] \mid \tilde{\mathfrak{F}}_{\mathbf{k}-1})} \exp \left\{ \frac{e^{\alpha h y - 1 - \alpha h y}}{y^r} E \left( |X_{\mathbf{k}}|^r I[0 < |X_{\mathbf{k}}| \leq y] \mid \tilde{\mathfrak{F}}_{\mathbf{k}-1} \right) + \right. \\
& \left. \alpha h E(X_{\mathbf{k}} I[|X_{\mathbf{k}}| \leq y] \mid \tilde{\mathfrak{F}}_{\mathbf{k}-1}) \right\} \leq \exp \left\{ \frac{e^{\alpha h y - 1 - \alpha h y}}{y^r} E \left( |X_{\mathbf{k}}|^r I[0 < |X_{\mathbf{k}}| \leq y] \mid \tilde{\mathfrak{F}}_{\mathbf{k}-1} \right) - \right. \\
& \left. \alpha h E(X_{\mathbf{k}} I[X_{\mathbf{k}} < -y] \mid \tilde{\mathfrak{F}}_{\mathbf{k}-1}) \right\} \leq \exp \left\{ \frac{e^{\alpha h y - 1 - \alpha h y}}{y^r} b_{\mathbf{k}}^r + \alpha h d_{\mathbf{k}} \right\}.
\end{aligned} \tag{34}$$

Now, furnishing  $\{\mathbf{k} : \mathbf{k} \leq \mathbf{n}\}$  with a total order and using property (F5), we have

$$E e^{\alpha h T_{\mathbf{n}}} \leq \exp \left\{ \frac{e^{\alpha h y - 1 - \alpha h y}}{y^r} \sum_{\mathbf{k} \leq \mathbf{n}} b_{\mathbf{k}}^r + \alpha h \sum_{\mathbf{k} \leq \mathbf{n}} d_{\mathbf{k}} \right\}. \tag{35}$$

Combining (31), (32), (33) and (35) we get

$$\begin{aligned}
& P(\max_{\mathbf{k} \leq \mathbf{n}} S_{\mathbf{k}} \geq x) \leq \\
& P(\max_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}} \geq y) + \left( \frac{\alpha}{\alpha - 1} \right)^{\alpha(d-1)} e^{-\alpha h x} \exp \left\{ \frac{e^{\alpha h y - 1 - \alpha h y}}{y^r} B_{\mathbf{k}}^r + \alpha h D_{\mathbf{k}} \right\} \leq \\
& P(\max_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}} \geq y) + e^{d-1} \exp \left\{ \frac{e^{\alpha h y - 1 - \alpha h y}}{y^r} B_{\mathbf{k}}^r + \alpha h D_{\mathbf{k}} - \alpha h x \right\}.
\end{aligned} \tag{36}$$

Setting,  $\alpha h = \frac{1}{y} \ln[\frac{xy^{r-1}}{B_{\mathbf{n}}^{r-1}} + 1]$  one can obtain (29).

To prove (30), we set:  $Y_{\mathbf{k}} = -X_{\mathbf{k}}$  and  $U_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} Y_{\mathbf{k}}$ . Obviously,  $\{(U_{\mathbf{n}}, \tilde{\mathfrak{F}}_{\mathbf{n}}), \mathbf{n} \in \mathbb{N}^d\}$  is martingale random field satisfying assumption of our theorem. Furthermore, denote  $\tilde{Y}_{\mathbf{k}} = Y_{\mathbf{k}} I[Y_{\mathbf{k}} \leq y]$  and  $\tilde{U}_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} \tilde{Y}_{\mathbf{k}}$ . By standard estimation we have

$$P(\max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{k}}| \geq x) \leq P(\max_{\mathbf{k} \leq \mathbf{n}} |X_{\mathbf{k}}| \geq y) + P(\max_{\mathbf{k} \leq \mathbf{n}} \widetilde{S}_{\mathbf{k}} \geq x) + P(\max_{\mathbf{k} \leq \mathbf{n}} \widetilde{U}_{\mathbf{k}} \geq x),$$

then similarly, as in the first part of the proof we obtain (30). ■

**Lemma 4.2.** *Let  $\{(X_{\mathbf{n}}, \mathfrak{F}_{\mathbf{n}}), \mathbf{n} \in \mathbb{N}^d\}$  satisfies assumption of Theorem 4.1, then there exist constant  $C > 0$  such that for every  $x > 0$  and  $j > 0$*

$$P(\max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{k}}| \geq x) \leq P(\max_{\mathbf{k} \leq \mathbf{n}} |X_{\mathbf{k}}| \geq \frac{x}{j}) + C \left( \frac{1}{x^r} \widetilde{M}_{\mathbf{n}}^r \right)^j.$$

The proof of this lemma is similar to those of Lemma 2.5, thus we omit it. Application of Lemma 4.2 gives the following two theorems. The first one is martingale random field version of Theorem 1.3 ((1.10)  $\Rightarrow$  (1.11)) of Gut et al. [6] and the latter, an extension of Theorem 1(b) and Theorem 2 of Ghosal and Chandra [5] to martingale random field with weaker moment restriction.

**Theorem 4.3.** *Let  $\{(X_{\mathbf{n}}, \mathfrak{F}_{\mathbf{n}}), \mathbf{n} \in \mathbb{N}^d\}$  satisfies assumption of Theorem 4.1 and WMD condition, moreover suppose that  $\alpha_1 > \frac{1}{2}$ ,  $\alpha_1 r > 1$  and there exist constant  $M$  depend only on  $r$  and  $\mathbf{n}_1 \in \mathbb{N}^d$  such that  $\frac{1}{|\mathbf{n}|} \widetilde{M}_{\mathbf{n}}^r \leq M$  for any  $\mathbf{n} \geq \mathbf{n}_1$ , thus if*

$$E | \xi |^r (\log_+ | \xi |)^{p-1} < \infty, \quad (37)$$

then

$$\sum_{\mathbf{n}} |\mathbf{n}|^{\alpha_1 r - 2} P(\max_{\mathbf{k} \leq \mathbf{n}} |S_{\mathbf{k}}| > |\mathbf{n}^{\alpha}| \varepsilon) < \infty \quad \text{for all } \varepsilon > 0. \quad (38)$$

**Proof.** Likewise the proof of (7)  $\Rightarrow$  (8). ■

Now, suppose that  $\{\mathbf{k}_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$  is a family of lattice points of  $\mathbb{N}^d$ .

**Theorem 4.4.** *Let  $\{X_{\mathbf{n},\mathbf{i}}, \mathbf{i} \leq \mathbf{k}_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$  be a  $d$ -dimensional array of row-wise martingale differences with respect to family of  $\sigma$ -algebras  $\{\mathfrak{F}_{\mathbf{n},\mathbf{i}}, \mathbf{i} \leq \mathbf{k}_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$  satisfying assumptions of Theorem 4.1. Furthermore, there exist families of non-negative constants  $\{a_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$  and  $\{\widetilde{M}_{\mathbf{k}_{\mathbf{n}}}^r, \mathbf{n} \in \mathbb{N}^d\}$  such that  $\sum_{\mathbf{j} \leq \mathbf{k}_{\mathbf{n}}} E(|X_{\mathbf{n},\mathbf{j}}|^r | \mathfrak{F}_{\mathbf{n},\mathbf{j}-1}) \leq \widetilde{M}_{\mathbf{k}_{\mathbf{n}}}^r$ . If the following conditions hold:*

- $\sum_{\mathbf{n}} a_{\mathbf{n}} P(\max_{\mathbf{i} \leq \mathbf{k}_{\mathbf{n}}} |X_{\mathbf{n},\mathbf{i}}| > \epsilon) < \infty$  for all  $\epsilon > 0$
- there exist  $j > 0$  such that  $\sum_{\mathbf{n}} a_{\mathbf{n}} \left(\widetilde{M}_{\mathbf{k}_{\mathbf{n}}}^r\right)^j < \infty$ ,

then

$$\sum_{\mathbf{n}} a_{\mathbf{n}} P\left(\max_{\mathbf{l} \leq \mathbf{k}_{\mathbf{n}}} \left| \sum_{\mathbf{i} \leq \mathbf{l}} X_{\mathbf{n},\mathbf{i}} \right| > \epsilon\right) < \infty \quad \text{for all } \epsilon > 0.$$

#### Negatively associated random fields - some comments.

Fuk-Nagaev inequality for sequences of negatively associated random variables can be proved by application of the comparison theorem which has been obtained by Shao [13].

**Theorem 4.5.** *Let  $\{X_i, 1 \leq i \leq n\}$  be a negatively associated sequence and let  $\{X_i^*, 1 \leq i \leq n\}$  be a sequence of independent random variables such that  $X_i^*$  and  $X_i$  have the same distribution for each  $i = 1, 2, \dots, n$ . Then*

$$Ef\left(\max_{1 \leq k \leq n} \sum_{i=1}^k X_i\right) \leq Ef\left(\max_{1 \leq k \leq n} \sum_{i=1}^k X_i^*\right) \quad (39)$$

for any convex and non-decreasing function  $f$  on  $\mathbf{R}^1$ , whenever the expectation on the right side exist.

In the case  $d \geq 2$ , Bulinski and Suquet (cf, Theorem 2.12 of [3]) have proved, that the comparison theorem does not hold in general for maximum of sums of NA random field.

**Theorem 4.6.** *Let  $f : \mathbf{R} \rightarrow \mathbf{R}$  be a function such that  $f(1) > f(0)$  (in particular, strictly increasing). Then, for any  $d > 1$  there exists a NA random field  $X = \{X_{\mathbf{j}}; \mathbf{j} \in Z^d\}$  and a multiindex  $\mathbf{n} \in N^d$  such that*

$$Ef \left( \max_{\mathbf{k} \leq \mathbf{n}} \sum_{\mathbf{i} \leq \mathbf{k}} X_{\mathbf{i}} \right) > Ef \left( \max_{\mathbf{k} \leq \mathbf{n}} \sum_{\mathbf{i} \leq \mathbf{k}} X_{\mathbf{i}}^* \right), \quad (40)$$

where  $X^* = \{X_{\mathbf{j}}^*; \mathbf{j} \in Z^d\}$  is decoupled version of  $X$ .

Furthermore, Shao has used in his proof, the maximal inequality for non-negative supermartingale which is not true for supermartingale random fields. Thus, the maximal Fuk-Nagaev inequality for NA random fields is open question.

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